Pi, Archimedes and circular splines

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Abstract

In the present paper, we give approximate values of π deduced from the *areas* of inscribed and circumscribed quadratic and cubic circular splines. Similar results on circular splines of higher degrees and higher approximation orders can be obtained in the same way. We compare these values to those obtained by computing the *perimeters* of those circular splines. We observe that the former are much easier to compute than the latter and give results of the same order. It also appears that Richardson extrapolation is very efficient on sequences of areas and give very good approximations of π. Finally, we also consider circular curves obtained by two subdivision algorithms.

1 Introduction: Archimedes and the computation of π

In his famous treatise *On the quadrature of the circle* [1], Archimedes (287-212 b.c.) gave the following upper and lower bounds for π

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$$

For this purpose, he computed the *perimeters* of inscribed and circumscribed polygons to the unit circle, starting from the regular hexagon and doubling the number of edges up to the regular polygon with 96 edges.

On the other hand, he also computed ([24], chapter 5) the *area of a sector of parabola* delimited by a secant and two tangents. From that result, he could have constructed the uniform piecewise parabolic circular curves which are tangent to the circle at the vertices or extreme points of parabolic arcs. From that he would have deduced approximations of π by computing the areas inside those closed curves. The latter are C^1 quadratic circular splines, and constitute a particular case of those described by Schoenberg in [5, 22, 23] and de Boor in [4].

In China, the mathematician Liu Hui (3rd century, see e.g. [7], [9], chapter 5, p. 173, [14, 25]), in his comments to the *Nine Chapters on the Mathematical Art* (Jiuzhang Suanshu, 200 b.c.), had the idea of computing the *areas* of regular inscribed and circumscribed polygons. He got the approximation 3.14 by using a regular polygon of 96 sides and the approximation 3.14159 by considering a polygon of 3072 sides.

In the present paper, we give approximate values of π deduced from the *areas* of inscribed and circumscribed quadratic and cubic circular splines. Similar results on circular splines of higher degrees (and higher approximation orders) can be obtained in the same way and will be published elsewhere. We compare these values to those obtained by computing the *perimeters* of those circular splines. We observe that the former are much easier to compute than the latter and give results of the same order. It also appears that Richardson extrapolation is very efficient on sequences of areas and give very good approximations of π. Finally, we also consider circular curves obtained by subdivision algorithms.

The paper is organised as follows. Section 2 is devoted to quadratic circular splines. We first give the equations of a parabolic arc tangent to the circle and we compute the areas and perimeters of the associated circular C^1 splines respectively outside and inside the unit circle. From that we deduce some upper and lower bounds on the value of π. A similar program is developed in Section 3 for C^2 circular cubic splines. In Section 4, Richardson extrapolation is applied to sequences of areas of interior and exterior circular quadratic and cubic splines. In Section 5, we study the approximation of the circle by de Rham curves and those obtained by two particular subfamilies of Merrien's subdivision algorithm.

It is clear that the results obtained by these methods cannot be compared with those obtained by powerful algorithms deduced from modular functions (see e.g. [2, 3, 6, 10, 12]). **Complete the conclusion.**

2 Quadratic circular splines

For $\theta = \pi/n, n \geq 3$, consider the C^1 piecewise quadratic (circular) spline whose spline polygon $\mathcal{S}_n$ is the regular polygon with n vertices $(\cos(k\theta), \sin(k\theta))$ tangent to the unit circle C..

2.1 Parabolic sector of the circular spline

One parabolic sector of angle 2θ, symmetric with respect to the x-axis, has a control polygon with the three vertices:

$$S_0 = (c, -s), \quad S_1 = (1/c, 0), \quad S_2 = (c, s), \quad \text{with } c := \cos(\theta),\ s := \sin(\theta)$$

The associated Bézier curve is

$$M(t) = (1-t)^2 S_0 + 2t(1-t) S_1 + t^2 S_2 \qquad t \in I := [0, 1],$$

whence the parametric equations of this arc of parabola

$$X(t) = c + 2(1/c - c)t(1-t) \qquad Y(t) = s(2t-1),$$

and the coordinates of its vertex $M(1/2)$

$$X_m = \frac{1}{2}(c + 1/c), \qquad Y_m = 0$$

Lemma 1. *This parabolic arc is exterior to the circle of radius $r_i = 1$ and tangent to it at its extreme points S_0 and S_2. Moreover, it lies inside the circle of radius $r_e := \frac{1}{2}(c + 1/c)$*

and is tangent to it at its vertex $M(1/2)$. Therefore, the quadratic circular spline lies between the two circles and is tangent to both of them.

Proof. The distance of $M(t)$ to the origin

$$OM(t)^2 = X(t)^2 + Y(t)^2 = (c + 2(1/c - c)t(1-t))^2 + s^2(2t-1)^2$$

simplifies to

$$OM(t)^2 = 1 + 4(1/c - c)^2 t^2 (1-t)^2 \geq 1$$

and is equal to 1 iff $t = 0, 1$. Moreover, its derivative being proportional to $2t(1-t)(1-2t)$, one deduces that the minimal distance holds for $t = 0, 1$ and the maximum one for $t = 1/2$, i.e. at the vertex where the curve is tangent to the circle of radius $r_e = \frac{1}{2}(c + 1/c)$ □.

2.2 Approximation of the area of the circle

The circle $C(r)$ lying between the two quadratic circular splines whose vertices are on circles $C(r_e)$ and C, its area is bounded by those of these splines. Let $t := \tan(\theta) = s/c$, then the area of the parabolic sector OS_0MS_2 is

$$a_e(\theta) = \frac{1}{2}\int_0^1 (XY' - X'Y)dt = \frac{1}{3}(c^2+2)t.$$

Therefore, the area of the parabolic sector inside $C(r)$ and tangent to it at its vertex is

$$a_i(\theta) = r_e^{-2} a_e(\theta) = \frac{4}{3}\frac{cs(c^2+2)}{(c^2+1)^2}$$

Therefore, the global areas of exterior and interior circular splines are given by

$$A_e(\theta) = na_e(\theta) = (\pi/\theta)a_e(\theta), \quad A_i(\theta) = na_i(\theta) = (\pi/\theta)a_i(\theta)$$

and their asymptotic expansions are respectively

$$A_e(\theta) = \pi\left(1 + \frac{2}{15}\theta^4 + \frac{2}{63}\theta^6 + O(\theta^8)\right)$$

$$A_i(\theta) = \pi\left(1 - \frac{7}{60}\theta^4 - \frac{13}{252}\theta^6 + O(\theta^8)\right)$$

One obtains immediately the extrapolated formula

$$A_m(\theta) = \frac{1}{15}(7A_e(\theta) + 8A_i(\theta)) = \frac{t}{45}\frac{(c^2+2)(7c^4+46c^2+7)}{(c^2+1)^2}$$

whose asymptotic expansion is

$$A_m(\theta) = \pi\left(1 - \frac{4}{315}\theta^6 + O(\theta^8)\right)$$

Example: for $n = 6$, $\theta = \pi/6, c = \sqrt{3}/2, s = 1/2, t = 1/\sqrt{3}$, one obtains

$$A_e(\pi/6) = 2t(c^2 + 2) = \frac{11}{6}\sqrt{3} \approx 3.1754$$

$$A_i(\pi/6) = 8\frac{cs(c^2 + 2)}{(c^2 + 1)^2} = \frac{88}{49}\sqrt{3} \approx 3.1106$$

$$A_m(\pi/6) = \frac{7997}{4410}\sqrt{3} \approx 3.1409$$

2.3 Approximation of the perimeter of the circle

The length of the parabolic arc with control polygon $S_0S_1S_2$ is given in [11] (p. ?). Denoting $|v|$ the euclidean norm of the vector v, one defines the vectors $v_0 := S_0S_1, v_1 := S_1S_2, v_2 := S_1S_2 - S_0S_1$, their norms

$$\nu_0 := |v_0|, \quad \nu_1 := |v_1|, \quad \nu_2 := |v_2|,$$

and their scalar products:

$$\sigma_{01} := \langle v_0, v_1\rangle, \quad \sigma_{02} = \langle v_0, v_2\rangle, \quad \sigma_{12} := \langle v_1, v_2\rangle.$$

The length of the parabolic arc is then given by the expression

$$\ell(\theta) = \frac{(\nu_0\nu_1)^2 - \sigma_{01}^2}{\nu_2^3} \ln\left(\frac{\sigma_{12} + \nu_1\nu_2}{\sigma_{02} + \nu_0\nu_2}\right) + \frac{\nu_1\sigma_{12} - \nu_0\sigma_{02}}{\nu_2^2}.$$

One deduces the length of the parabolic arc outside the unit circle

$$\ell_e(\theta) = t - \frac{c}{2}\ln\left(\frac{1-s}{1+s}\right), \quad t = \tan(\theta)$$

and that of the parabolic arc inside the unit circle

$$\ell_i(\theta) = \frac{2c}{1+c^2}\ell_e(\theta).$$

As $n = \frac{\pi}{\theta}$, the perimeters of the associated circular quadratic splines are respectively

$$P_e(\theta) = \frac{\pi}{\theta}\ell_e(\theta), \qquad P_i(\theta) = \frac{\pi}{\theta}\ell_i(\theta)$$

and their asymptotic expansions are respectively

$$P_e(\theta) = 2\pi(1 + \frac{1}{15}\theta^4 + \frac{8}{315}\theta^6 + O(\theta^8))$$

$$P_i(\theta) = 2\pi(1 - \frac{7}{120}\theta^4 - \frac{41}{2520}\theta^6 + O(\theta^8))$$

One obtains immediately the extrapolated formula

$$P_m := \frac{1}{15}(7P_e(\theta) + 8P_i(\theta))$$

whose asymptotic expansion is

$$P_m(\theta) = 2\pi\left(1 + \frac{1}{315}\theta^6 + O(\theta^8)\right)$$

Example: for $n = 6$, $\theta = \pi/3, s = \sqrt{3}/2, c = 1/2, t = \sqrt{3}$, one obtains

$$\frac{1}{2}P_i(\pi/6) = \frac{3}{7}(4 + 3\ln(3)) \approx 3.126$$

$$\frac{1}{2}P_e(\pi/3) = \frac{\sqrt{3}}{4}(4 + 3\ln(3)) \approx 3.159$$

$$\frac{1}{2}P_m(\pi/3) = \frac{32}{35} + \frac{7}{15}\sqrt{3} + \left(\frac{24}{35} + \frac{7}{20}\sqrt{3}\right)\ln(3) \approx 3.1419$$

Finally

$$AP_m := \frac{1}{5}(A_m + 4P'_m) = \frac{1}{5}\left(\frac{128}{35} + \frac{16229}{4410}\sqrt{3} + \left(\frac{96}{35} + \frac{28}{20}\sqrt{3}\right)\ln(3)\right) \approx 3.1417$$

2.4 Approximate values of π deduced from the areas of quadratic circular splines

The following table gives the errors on π deduced from preceding formulas, for some values of n. We use the notation $P'_e := \frac{1}{2}P_e$ (resp. $P'_i := \frac{1}{2}P_i$, $P'_m := \frac{1}{2}P_m$). Moreover, comparing the expansions of P'_m and A_m

$$A_m(\theta) = \pi\left(1 - \frac{4}{315}\theta^6 + O(\theta^8)\right), \quad P'_m = \pi\left(1 + \frac{2}{945}\theta^6 + O(\theta^8)\right)$$

one obtains the linear combination

$$AP_m := \frac{1}{5}(A_m + 4P'_m) = \pi\left(1 + \frac{157}{31500}\theta^8 + O(\theta^{10})\right)$$

n	$A_e - \pi$	$P'_e - \pi$	$\pi - P'_i$	$\pi - A_i$	$A_m - \pi$	$P'_m - \pi$	$\pi - AP_m$
6	3.4(-2)	1.7(-2)	1.5(-2)	3.1(-2)	7.3(-4)	3.1(-4)	**1.1(−4)**
12	2.0(-3)	1.0(-3)	8.8(-4)	1.8(-3)	1.2(-5)	3.6(-6)	**3.6(−7)**
24	1.2(-4)	6.2(-5)	5.4(-5)	1.1(-4)	2.0(-7)	5.2(-8)	**1.4(−9)**
48	7.7(-6)	3.8(-6)	3.4(-6)	6.7(-6)	3.1(-9)	7.9(-10)	**5.3(−12)**
96	4.8(-7)	2.4(-7)	2.1(-7)	4.2(-7)	4.9(-11)	1.2(-11)	**2.1(−14)**
192	3.0(-8)	1.5(-8)	1.3(-8)	2.8(-8)	7.6(-13)	1.9(-13)	**8.0(−17)**
384	1.9(-9)	9.4(-10)	8.2(-10)	1.6(-9)	1.2(-14)	3.0(-15)	**3.1(−19)**
768	1.2(-10)	5.9(-11)	5.1(-11)	1.0(-10)	1.9(-16)	4.7(-17)	**1.2(−21)**
1536	7.3(-12)	3.7(-12)	3.2(-12)	6.4(-12)	2.9(-18)	7.3(-19)	**4.8(−24)**

For $\mathbf{n = 96}$ (the value used by Archimedes), one gets 6 exact digits

$$A_i(\pi/96) \approx 3.14159223 < \pi < A_e(\pi/96) \approx 3.14159313$$

while Archimedes got only 3 exact digits with

$$3 + \frac{10}{71} \approx 3.1408 < \pi < \frac{22}{7} \approx 3.1428$$

Moreover, the extrapolated values give 11 exact digits for A_m

$$A_m(\pi/96) \approx \mathbf{3.14159265354}081$$

and 10 digits for P_m

$$P_m(\pi/96) \approx \mathbf{3.1415926536}020$$

For $\mathbf{n} = \mathbf{1536}$, one gets 18 exact digits with A_m and P_m:

$$A_m(\pi/1536) \approx \mathbf{3.141592653589793235}542$$

$$P_m(\pi/1536) \approx \mathbf{3.141592653589793239}192$$

and the final linear combination gives 24 exact digits:

$$AP_m(\pi/1536) \approx \mathbf{3.141592653589793238462648}178$$

3 Cubic circular splines

Consider the C^2 piecewise cubic (circular) spline whose spline polygon $\mathcal{S}_n$ is the regular polygon with n vertices tangent to the circle $C(r)$ with radius r.

3.1 Equations of the cubic arc

We use the notations: $\theta = \pi/n, c := \cos(\theta), s := \sin(\theta), c_k := \cos(k\theta), s_k := \sin(k\theta), k = 2, 3$. Given a radius $r > 0$, consider the cubic sector of angle 2θ, symmetric with respect to the x-axis whose spline vertices are

$$S_0 = r(c_3, -s_3), \quad S_1 := r(c, -s), \quad S_2 = r(c, s), \quad S_3 = r(c_3, s_3)$$

In the article [20], the author designed an algorithm, called the SB-algorithm, giving the control vertices of the local Bézier form of the cubic arc in function of its spline vertices. One first construct the four points

$$B_0 := \frac{1}{3}(S_0 + 2S_1) = r\left(\frac{1}{3}(c_3 + 2c), -\frac{1}{3}(s_3 + 2s)\right),$$

$$B_1 := \frac{1}{3}(2S_1 + S_2) = r(c, -s/3),$$

$$B_2 := \frac{1}{3}(S_1 + 2S_2) = r(c, s/3),$$

$$B_3 := \frac{1}{3}(2S_2 + S_3) = r\left(\frac{1}{3}(c_3 + 2c), \frac{1}{3}(s_3 + 2s)\right)$$

From them, one deduces the vertices of the local B-polygon

$$C_0 = \frac{1}{2}(B_0 + B_1) = r\left(\frac{1}{6}(c_3 + 5c), -\frac{1}{6}(s_3 + 3s)\right)$$
$$C_1 = B_1, \quad C_2 = B_2,$$
$$C_3 = \frac{1}{2}(B_2 + B_3) = r\left(\frac{1}{6}(c_3 + 5c), \frac{1}{6}(s_3 + 3s)\right)$$

The associated Bézier curve is then

$$M(u) := C_0\,(1-u)^3 + 3C_1\,u(1-u)^2 + 3C_2\,u^2(1-u) + C_3\,u^3, \quad u \in [0,1]$$

Its parametric equations are respectively, after simplification

$$X(u) = r\left(\frac{1}{3}(2c^2+1) + 2s^2u(1-u)\right)$$
$$Y(u) = r\left(\frac{1}{3}t(2u-1)(2c^2+1+2s^2u(1-u))\right)$$

3.2 Exterior and interior circles

For $u = 1/2$, we get the coordinates of the midpoint of the cubic arc

$$X_m = \frac{r}{6}(c^2+5), \qquad Y_m = 0$$

The radii of two circles appear. The first one, r_e, is that of the circle passing through C_0 and C_3 and tangent to the cubic at those points ($t = s/c = \tan(\theta)$)

$$r_e^2 = \frac{r^2}{9}(2c^2+1)^2(1+t^2) \rightarrow r_e = \frac{r}{3c}\left(2c^2+1\right)$$

The second one, r_i, is that of the circle through $M(1/2)$ and tangent to the cubic at that point

$$r_i = X_m = \frac{r}{6}(c^2+5).$$

Comparing the two radii shows that we have always

$$r_e - r_i = \frac{r}{6c}(1-c)^2(2-c) > 0$$

and that this difference tends to zero when $\theta \rightarrow 0$.

Lemma 2. *The cubic arc is interior to the circle of radius r_e and tangent to it at the extreme points C_0 and C_3. Moreover, it lies outside the circle of radius r_i and is tangent to it at its midpoint $M(1/2)$.*
Proof.

$$d(u)^2 = r^2\left(\frac{1}{3}(2c^2+1) + 2s^2u(1-u)\right)^2 + r^2\left(\frac{1}{3}t(2u-1)(2c^2+1+2s^2u(1-u))\right)^2$$

The numerator of the derivative is proportional to the polynomial

$$p_5(u) := u(1-u)(2u-1)(1+2s^2u(1-u))$$

We see that it has the sign of $u(1-u)(2u-1)$, i.e. $d(u)$ is decreasing for $u \in [0,1/2]$ and increasing for $u \in [1/2,1]$. $\square$.

3.3 Approximation of the area of the circle

The cubic circular spline lying between the two circles $C(r_e)$ and $C(r_i)$, its area is bounded by those of these circles. Conversely, choosing $r_e = 1$, i.e. $r = \frac{3c}{2c^2+1}$, the unit circle is tangent at the extreme points of the corresponding cubic arc and also at the midpoint of another cubic arc exterior to it.
Therefore, the area of the cubic sector inside $C(1)$ and tangent to it at its extreme points is

$$a_i(\theta) = \frac{1}{2}r^2 \int_0^1 (XY' - X'Y)du$$

The global areas are respectively

$$A_e = \frac{4\pi}{5\theta} \frac{s(2c^4 + 26c^2 + 17)}{c(c^4 + 10c^2 + 25)}, \qquad A_i = \frac{\pi}{5\theta} \frac{sc(2c^4 + 26c^2 + 17)}{c^4 + 4c^2 + 1}$$

Their asymptotic expansions are respectively

$$A_e(\theta) = \pi \left(1 + \frac{7}{180}\theta^4 + \frac{31}{756}\theta^6 + O(\theta^8)\right)$$

$$A_i(\theta) = \pi \left(1 - \frac{2}{45}\theta^4 - \frac{8}{189}\theta^6 + O(\theta^8)\right)$$

The extrapolated formula is then

$$A_m(\theta) = (8A_e + 7A_i)/15 = \pi \left(1 + \frac{2}{945}\theta^6 + O(\theta^7)\right)$$

3.4 Approximation of the perimeter of the circle

Computing the length of the cubic arc, i.e. $\ell(\theta) = \int_0^1 \sqrt{X'(u)^2 + Y'(u)^2}\, du$ gives, after simplificaion

$$\ell_e(\theta) = 2sc \int_0^1 \sqrt{1 + 4s^2t^2u^2(1-u)^2}\, du$$

This integral can be computed by expanding the function $f(v) = \sqrt{1 + v^2}$ in power of $v := 2stu(1-u) \leq \frac{1}{2}st < 1$ for $st < 2$, and to integrate it term by term. We first have

$$f(v) = 1 + \frac{1}{2}v^2 + \sum_{k\geq 2} \left(-\frac{1}{4}\right)^{k-1} \binom{2k-3}{k-2} \frac{v^{2k}}{k}$$

Then, using the integrals

$$\int_0^1 (u(1-u))^{2k}\, du = \beta(2k+1, 2k+1) = \frac{\Gamma(2k+1)^2}{\Gamma(4k+2)} = \frac{(2k!)^2}{(4k+1)!} = \frac{1}{4k+1} \binom{4k}{2k}^{s-1}$$

we get

$$\ell_e(\theta) = 2sc \left(1 + \frac{1}{15}s^2t^2 + 4\sum_{k\geq 2} (-1)^{k-1} \binom{2k-3}{k-2} \binom{4k}{2k}^{-1} \frac{(st)^{2k}}{k(4k+1)}\right)$$

$$= 2sc\left(1 + \frac{1}{15}s^2t^2 - \frac{1}{315}s^4t^4 + \frac{1}{3003}s^6t^6 + \ldots\right)$$

On the other hand, we have

$$\ell_i(\theta) = \frac{r_1}{r_0}\ell_e = \frac{c(c^2+5)}{2(c^2+1)}\ell_e$$

The perimeters of the circular cubic splines being respectively:

$$P_e(\theta) = n\ell_e(\theta), \quad P_i(\theta) = n\ell_i(\theta),$$

we obtain their asymptotic expansions

$$P_e(\theta) = 2\pi\left(1 + \frac{7}{360}\theta^4 + \frac{163}{7560}\theta^6 + O(\theta^8)\right)$$

$$P_i(\theta) = 2\pi\left(1 - \frac{1}{15}\theta^4 - \frac{19}{945}\theta^6 + O(\theta^8)\right)$$

and the extrapolated formula is then

$$P_m(\theta) = (8P_e + 7APi)/15 = \pi\left(1 + \frac{2}{945}\theta^6 + O(\theta^7)\right).$$

3.5 Approximations of π

The following table gives the errors on π deduced from the preceding formulas A_e, A_i, A_m and P'_e, P'_i, P'_m (half perimeters) for some values of n

n	$A_e - \pi$	$P'_e - \pi$	$\pi - P'_i$	$\pi - A_i$	$A_m - \pi$	$P'_m - \pi$
6	1.2(-2)	6.2(-3)	6.6(-3)	1.3(-2)	1.6(-4)	1.8(-4)
12	6.2(-4)	3.1(-4)	3.5(-4)	7.0(-4)	2.2(-6)	2.3(-6)
24	3.6(-5)	1.8(-5)	2.1(-5)	4.2(-5)	3.4(-8)	3.4(-8)
48	2.2(-6)	1.1(-6)	1.3(-6)	2.6(-6)	5.2(-10)	5.2(-10)
96	1.4(-7)	7.0(-8)	8.0(-8)	1.6(-7)	8.2(-12)	8.2(-12)
192	8.8(-9)	4.4(-9)	5.0(-9)	1.0(-8)	1.3(-13)	1.3(-13)
384	5.5(-10)	2.7(-10)	3.1(-10)	6.2(-10)	2.0(-15)	2.0(-15)
768	3.4(-11)	1.7(-11)	1.9(-11)	3.9(-11)	3.1(-17)	3.1(-17)
1536	2.1(-12)	1.0(-12)	1.2(-12)	2.4(-12)	4.9(-19)	4.9(-19)

Unfortunately, the signs of the errors $A_m - \pi$ and $P'_m - \pi$ are the same and no extrapolation formula is possible. A better result is obtained via Richardson extrapolation.

4 Richardson extrapolation

4.1 Quadratic splines

One can extrapolate the values $A_e(n)$ by using the Richardson's extrapolation algorithm (see e.g. [8], chapter 2). The linear combination $AR_e(\theta) = (16A_e(\theta/2) - A_e(\theta))/15$ gives

$$AR_e(\theta) = \pi\left(1 - \frac{1}{630}\theta^6 + O(\theta^8)\right)$$

The order is the same as for $A_m(\theta)$, however the latter is slightly better.

n	$A_e - \pi$	$\pi - A_i$	$\pi - A_m$	$\pi - AR_e$	$AR_i - \pi$	$\pi - AR$
12	2.0(-3)	1.8(-3)	1.2(-5)	1.7(-6)	2.6(-6)	**3.1(−8)**
24	1.2(-4)	1.1(-4)	2.0(-7)	2.5(-8)	4.1(-8)	**1.2(−10)**
48	7.7(-6)	6.7(-6)	3.1(-9)	3.9(-10)	6.4(-10)	**4.6(−13)**
96	4.8(-7)	4.2(-7)	4.9(-11)	6.1(-12)	9.9(-12)	**1.8(−15)**
192	3.0(-8)	2.6(-8)	7.6(-13)	9.6(-14)	1.5(-13)	**6.9(−18)**
384	1.9(-9)	1.6(-9)	1.2(-14)	1.5(-15)	2.4(-15)	**2.7(−20)**
768	1.2(-10)	1.0(-10)	1.9(-16)	2.3(-17)	3.8(-17)	**1.0(−22)**
1536	7.3(-12)	6.4(-12)	2.9(-18)	3.6(-19)	5.9(-19)	**4.1(−25)**

For $n = 1536$, the extrapolated value gives 17 exact digits

$$AR_e(\pi/1536) \approx 3.141592653589793215$$

In the same way, for $A_i(\theta)$, the linear combination $AR_i(\theta) = (16A_i(\theta/2) - A_i(\theta))/15$ gives

$$AR_i(\theta) = \pi \left(1 + \frac{13}{5040}\theta^6 + O(\theta^8)\right)$$

The order is the same as for $A_m(\theta)$, however the latter is slightly better. Moreover, the linear combination $AR := \frac{13}{21}AR_e + \frac{8}{21}AR_i$ gives

$$\pi - AR = \frac{13}{30240}\theta^8 + O(\theta^{10})$$

For $n = 1536$, we obtain 25 exact digits.

4.2 Cubic splines

$$A_e(\theta) = \pi \left(1 + \frac{7}{180}\theta^4 + \frac{31}{756}\theta^6 + O(\theta^8)\right)$$

The linear combination $AR_e(\theta) = (16A_e(\theta/2) - A_e(\theta))/15$ gives

$$AR_e(\theta) = \pi \left(1 - \frac{31}{15120}\theta^6 + O(\theta^8)\right)$$

Notice that

$$A_m(\theta) - A_R(\theta) = \frac{\pi}{240}\theta^6 + O(\theta^8)$$

Now, extrapolating the values $A_i(\theta)$ by using the Richardson's extrapolation algorithm, the linear combination $AR_i(\theta) = (16A_i(\theta/2) - A_i(\theta))/15$ gives

$$AR_i(\theta) = \pi \left(1 + \frac{2}{945}\theta^6 + O(\theta^8)\right)$$

which has the same term in θ^6 as $A_m(\theta)$.

Finally, the linear combination $AR := (32AR_e + 31AR_i)/63$ gives the excellent approximation

$$\pi - AR = \frac{47\pi}{831600}\theta^{10} + O(\theta^{12})$$

n	$A_e-\pi$	$\pi-A_i$	$\pi-A_m$	$\pi-AR_e$	$AR_i-\pi$	$\pi-AR$
12	6.2(-4)	7.0(-4)	2.2(-6)	2.1(-6)	2.2(-6)	$\mathbf{2.9(-10)}$
24	3.6(-5)	4.2(-5)	3.4(-8)	3.3(-8)	3.4(-8)	$\mathbf{2.7(-13)}$
48	2.2(-6)	2.6(-6)	5.2(-10)	5.1(-10)	5.2(-10)	$\mathbf{2.6(-16)}$
96	1.4(-7)	1.6(-7)	8.2(-12)	7.9(-12)	8.2(-12)	$\mathbf{2.5(-19)}$
192	8.8(-9)	1.0(-8)	1.3(-13)	1.2(-13)	1.3(-13)	$\mathbf{2.4(-22)}$
384	5.5(-10)	6.2(-10)	2.0(-15)	1.9(-15)	2.0(-15)	$\mathbf{2.4(-25)}$
768	3.4(-11)	3.9(-11)	3.1(-17)	3.0(-17)	3.1(-17)	$\mathbf{2.3(-28)}$
1536	2.1(-12)	2.4(-12)	4.9(-19)	4.7(-19)	4.9(-19)	$\mathbf{2.3(-31)}$

4.3 Quadratic and cubic splines: complete extrapolation

4.3.1 Quadratic splines

Richardson extrapolation on the sequence A_e

1.2(-4)							
1.7(-6)	-2.3(-7)						
2.5(-8)	-7.9(-10)	9.6(-11)					
3.9(-10)	-3.0(-12)	8.3(-14)	-1.0(-14)				
6.1(-12)	-1.2(-14)	7.9(-17)	-2.3(-18)	2.8(-19)			
9.6(-14)	-4.5(-17)	7.7(-20)	-5.4(-22)	1.5(-23)	-1.9(-24)		
1.5(-15)	-1.8(-19)	7.5(-23)	-1.3(-25)	9.1(-28)	-2.6(-29)	$\mathbf{3.2(-30)}$	
2.3(-17)	-6.9(-22)	7.3(-26)	-3.2(-29)	5.5(-32)	-3.8(-34)	$\mathbf{1.1(-35)}$	$\mathbf{-1.4(-36)}$
1	2	3	4	5	6	7	8

Richardson extrapolation on the sequence A_i

-1.7(-4)							
-2.6(-6)	5.6(-8)						
-4.1(-8)	2.9(-10)	7.1(-11)					
-6.4(-10)	1.2(-12)	6.0(-14)	-9.2(-15)				
-9.9(-12)	4.7(-15)	5.6(-17)	-2.2(-18)	8.2(-20)			
-1.5(-13)	1.8(-17)	5.4(-20)	-5.2(-22)	5.7(-24)	7.0(-25)		
-2.4(-15)	7.2(-20)	5.3(-23)	-1.3(-25)	3.6(-28)	9.0(-30)	$\mathbf{-1.7(-30)}$	
-3.8(-17)	2.8(-22)	5.2(-26)	-3.1(-29)	2.2(-32)	1.3(-34)	$\mathbf{-6.3(-36)}$	$\mathbf{3.0(-37)}$
1	2	3	4	5	6	7	8

Richardson extrapolation on the sequence A_m

1.2(-6)							
3.4(-9)	-1.2(-9)						
1.2(-11)	-1.1(-12)	1.5(-13)					
4.9(-11)	4.7(-14)	-1.0(-15)	3.3(-17)	-2.6(-18)			
1.8(-16)	-9.9(-19)	8.0(-21)	-1.5(-22)	7.8(-24)			
7.1(-19)	-9.6(-22)	1.9(-24)	-9.2(-27)	1.1(-28)	-8.9(-30)		
2.8(-21)	-9.4(-25)	4.7(-28)	-5.6(-31)	1.6(-33)	-2.7(-35)	**7.2(−36)**	
1.1(-23)	-9.2(-28)	1.1(-31)	-3.4(-35)	2.5(-38)	-9.7(-41)	**5.7(−42)**	**−1.2(−42)**
1	2	3	4	5	6	7	8

4.3.2 Cubic splines

Richardson extrapolation on the sequence A_e

1.5(-4)							
2.1(-6)	-2.1(-7)						
3.2(-8)	-7.6(-10)	6.9(-11)					
5.1(-10)	-2.9(-12)	5.9(-14)	-8.0(-15)				
7.9(-12)	-1.1(-14)	5.6(-17)	-1.7(-18)	2.4(-19)			
1.2(-13)	-4.4(-17)	5.4(-20)	-4.0(-22)	1.3(-23)	-1.6(-24)		
1.9(-15)	-1.7(-19)	5.3(-23)	-9.8(-26)	7.7(-28)	-2.2(-29)	**2.6(−30)**	
3.0(-17)	-6.7(-22)	5.2(-26)	-2.4(-29)	4.7(-32)	-3.2(-34)	**8.9(−36)**	**−1.1(−36)**
1	2	3	4	5	6	7	8

Richardson extrapolation on the sequence A_i

-1.5(-4)							
-2.2(-6)	2.0(-8)						
-3.4(-8)	7.7(-10)	-3.1(-11)					
-5.2(-10)	3.0(-12)	-3.4(-14)	-3.4(-15)				
-8.2(-12)	1.2(-14)	-3.4(-17)	-6.4(-19)	2.0(-19)			
-1.3(-13)	4.5(-17)	-3.3(-20)	-1.4(-22)	1.1(-23)	-1.3(-24)		
-2.0(-15)	1.8(-19)	-3.2(-23)	-3.5(-26)	6.5(-28)	-1.9(-29)	**9.6(−31)**	
-3.1(-17)	6.9(-22)	-3.2(-26)	-8.5(-30)	3.9(-32)	2.8(-34)	**3.9(−36)**	**2.2(−37)**
1	2	3	4	5	6	7	8

Richardson extrapolation on the sequence A_m

2.6(-7)							
6.9(-10)	-3.3(-10)						
2.5(-12)	-2.4(-13)	8.8(-4)					
9.4(-15)	-2.1(-16)	1.8(-17)	-3.3(-18)				
3.6(-17)	-2.0(-19)	4.3(-21)	-1.8(-22)	2.2(-23)			
1.4(-19)	-2.0(-22)	1.0(-24)	-1.1(-26)	3.0(-28)	-2.8(-29)		
5.6(-22)	-1.9(-25)	2.5(-28)	-6.5(-31)	4.5(-33)	-9.8(-35)	**7.4(−36)**	
2.2(-24)	-1.9(-28)	6.1(-32)	-4.0(-35)	6.9(-38)	-3.7(-40)	**6.5(−42)**	**−5.0(−43)**
1	2	3	4	5	6	7	8

Remark: *instead of applying Richardson Algorithm (RA) to the sequence AR_m, we also may do a linear combination on columns of A_e and A_i with the same number in the RA.*

For example for cubics, we have

$$A_e(\theta) = \pi\left(1 + \frac{7}{180}\theta^4 + \frac{31}{756}\theta^6 + O(\theta^8)\right)$$

$$A_i(\theta) = \pi\left(1 - \frac{2}{45}\theta^4 - \frac{8}{189}\theta^6 + O(\theta^8)\right)$$

The first step of RE gives

$$A_e[1] = \pi\left(1 - \frac{31}{15120}\theta^6 + O(\theta^8)\right)$$

$$A_i[1] = \pi\left(1 + \frac{2}{945}\theta^6 + O(\theta^8)\right)$$

Thus, the first linear combination is

$$A_m[1] := (32A_e[1] + 31A_i[1])/63 = \pi\left(1 - \frac{47}{831600}\theta^{10} + O(\theta^{12})\right)$$

5 Related topics

5.1 De Rham circular splines for the approximation of the area

In some sense, de Rham splines [11] are generalizations of quadratic splines. For the approximation of the circle, one starts from the n-regular polygon $\mathcal{P}_0$ circumscribing the unit circle. Its vertices can be chosen as the points $P_k = r(\cos(k\theta), \sin(k\theta))$ where $\theta := 2\pi/n$, $r = 1/\cos(\theta)$ and $k = 0, 1, \ldots, n-1$. Then, given a parameter $\gamma > 1$, one defines the new polygon $\mathcal{P}_1$ whose $2n$ vertices $R_p, 0 \leq p \leq 2n-1$ are defined by the formulas

$$R_{2\ell} := \frac{\gamma+1}{\gamma+2}P_\ell + \frac{1}{\gamma+2}P_{\ell+1}$$

$$R_{2\ell+1} := \frac{1}{\gamma+2}P_\ell + \frac{\gamma+1}{\gamma+2}P_{\ell+1}$$

for $\ell = 0, \dots n-1$. The next polygon $\mathcal{P}_2$ is deduced from $\mathcal{P}_1$ in the same way, and more generally $\mathcal{P}_{m+1}$ is deduced from $\mathcal{P}_m$ by using the same construction. Let A_0 be the area of $\mathcal{P}_0$ and A_1 be that of $\mathcal{P}_1$. Then one can prove that the sequence of polygons $\{\mathcal{P}_m, m \in \mathbb{N}\}$ converges to a convex body $\mathcal{P}^*$ whose area is given by

$$A^* = A_0 + (A_1 - A_0)/(1-\bar{\gamma}), \quad \text{where} \quad \bar{\gamma} := \frac{2\gamma}{(\gamma+2)^2}$$

Here are the areas obtained for some values of parameters γ and for some values of n.

n	4	8	16	32	64	128	256
$\gamma = 1.8$	3.26	3.134	3.138	3.1405	3.1413	3.141525	3.141575
$\gamma = 1.9$	3.30	3.143	3.140	3.1411	3.14146	3.141560	3.141584
$\gamma = 2$	3.33	3.15	3.142	3.1416	**3.14159**5	**3.1415928**0	**3.1415926**6
$\gamma = 2.1$	3.36	3.159	3.144	3.1421	3.1417	3.141623	3.141600
$\gamma = 2.2$	3.39	3.167	3.146	3.1426	3.1418	3.141651	3.141607

It seems the best values obtained for $\gamma = 2$, i.e. for the quadratic spline.
One may also define a sequence of polygons inscribed in the unit circle.

5.2 Merrien's subdivision algorithm for the approximation of the area

The general C^1 Hermite subdivision scheme introduced by J.L. Merrien [16] is defined as follows. Given two abscissae $a < b$, $m := \frac{1}{2}(a+b)$ and $h := b - a$, the scheme depends on two negative parameters α, β. Starting from the values of a function f and its slope p at points a and b, one evaluates their values at the midpoint m:

$$f(m) := \frac{1}{2}(f(b) + f(a)) + \alpha h(p(b) - p(a))$$

$$p(m) := (1-\beta)\frac{1}{h}(f(b) - f(a)) + \beta\frac{1}{2}(p(b) + p(a))$$

At step 2, the process is then repeated in subintervals $[a, m]$ and $[m, b]$. More generally, at step k, the values of f and p are computed at midpoints of 2^{k-1} subintervals of $[a, b]$. Thus, when k tends to infinity, f and p are defined on the set $\mathcal{D}$ of dyadic points of $[a, b]$. One then proves that there exist a unique C^1 function f such that the restrictions of f and f' to $\mathcal{D}$ coincide with the values provided by the subdivision algorithm. Here, we start from the vertices of the regular n-gon inscribed in the unit circle and from the tangent vectors to that circle. Then the new vertices and tangent vectors are computed by using the preceding subdivision scheme. At step k, we get a (non regular) polygon with $n \cdot 2^{k-1}$ vertices and tangent vectors. Then, we consider the Hermite cubic spline interpolating those vertices and having those tangent vectors, and we compute the area of the body obtained in that way. The calculations are based on the following formula for the area of a cubic arc. Assume that this arc has endpoints $S_0 = (x_0, y_0)$ and $S_1 = (x_1, y_1)$ with the associated tangent vectors $T_0 = (p_0, q_0)$ and $T_1 = (p_1, q_1)$. Thus it can be written in the Bernstein-Bézier form as follows

$$M(u) = S_0(1-u)^3 + (3S_0 + hT_0)1u(1-u)^2 + (3S_1 - hT_1)u^2(1-u) + S_1u^3$$

Then a direct computation of the area of the cubic sector OS_0S_1 gives the formula

$$A(S_0S_1) = \frac{1}{2}(x_0y_1 - x_1y_0) + \frac{h}{10}((x_1 - x_0)(q_1 - q_0) - (y_1 - y_0)(p_1 - p_0)) + \frac{h^2}{60}(p_1q_0 - p_0q_1)$$

The global area is then obtained by summing up all these local areas.

Here are the results for the two families of parameters described in [17] and [18] respectively.

1)The first family is that of pairs (α, β) with $\beta \in [-1, 0)$ and $\alpha = \frac{\beta}{4(1-\beta)}$.
2) The second one is that of pairs (α, β) with $\beta \in [-1, 0)$ and $\alpha = \frac{\beta(\beta+2)}{4(1-\beta)}$.
For $\beta = -1$, both families give $\alpha = -\frac{1}{8}$ (C^1 quadratic splines) and for $\beta = -\frac{1}{2}$, the second family gives $\alpha = -\frac{1}{8}$ (C^1 cubic splines).

5.2.1 Results for the first family

n	4	8	16	32	64	128	256
$\beta = -0.9$	3.04	3.122	3.137	3.1405	3.1413	3.141526	3.141576
$\beta = -0.95$	3.07	3.131	3.139	3.1410	3.1414	3.141560	3.141584
$\beta = -1$	3.10	3.139	3.1414	3.141583	**3.141592**04	**3.14159261**1	**3.1415926512**
$\beta = -1.05$	3.13	3.147	3.1434	3.1420	3.1417	3.141623	3.141600
$\beta = -1.1$	3.16	3.154	3.1453	3.1425	3.1418	3.141653	3.141607

In that family, C^1 quadratic Hermite splines give the best approximate values of π.

5.2.2 Results for the second family

n	4	8	16	32	64	128	256
$\beta = -1/2$	3.092	3.1383	3.14138	3.141579	**3.14159**184	**3.1415926**03	**3.141592650**4
$\beta = -5/8$	3.162	3.1566	3.1460	3.1427	3.1419	3.14166	3.141610
$\beta = -3/4$	3.181	3.1612	3.1471	3.1430	3.1419	3.14168	3.141615
$\beta = -7/8$	3.159	3.1547	3.1454	3.1426	3.1418	3.14165	3.141608
$\beta = -1$	3.103	3.1391	3.14143	3.141583	**3.141592**04	**3.1415926**15	**3.141592651**2

In that family, C^1 quadratic and cubic Hermite splines give the best values of π.

6 Conclusion

Some ideas:
1) Curves obtained by subdivision do not give very interesting results.

2) Quadratics are as good as cubics. It is also true for higher degrees: the results for each degree pair $(4,5),(6,7),(8,9)$ are of the same approximation order, respectively $O(n^{-?}),O(n^{-?}),O(n^{-?}$
3) as said in the introduction, not comparable with strong modular algorithms like those of Ramanujan or Borwein [3, 6, 12]
4) maybe some good approximations of π by algebraic numbers could be obtained by the algorithms.

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